

Electromagnetic form factors of pion and rho in the three forms of relativistic kinematics

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Abstract

The electromagnetic form factors of the π and the ρ are obtained using the three forms of relativistic kinematics, instant form, point form and (light) front form. Simple representations of the mass operator together with single quark currents are employed with all the forms. The Poincaré covariant current operators are generated by the dynamics from single-quark currents that are covariant under the kinematic subgroup. Front and instant forms allow to reproduce the available data for the pion form factor. On the other hand point form is not able to reproduce qualitatively the experimental data with reasonable values for the wave function parameters. For the ρ electromagnetic form factors, instant and front forms provide a consistent picture. The obtained results do not depend appreciably on the wave function used.

The electromagnetic form factors of hadrons are an important source of information about their internal structure. They provide a useful tool to understand the dynamics of the strong interaction and the role played by relativity in understanding the transition region between the low-energy and perturbative QCD domains.

In the literature, there are several works where the form factors of the π and the ρ have been studied making use of relativistic quark models, e.g. Refs. [1,2,3,4,5,6]. Most theoretical studies were carried out making use of front form while only lately point form was also employed giving rise to some discrepancies in its formulation [4,5]. Here we present a comparative study of the form factors obtained with the three forms of relativistic kinematics making use of the same assumptions for the mass operator and the structure of the electromagnetic current. The understanding of the different formulations of relativistic quark models and their ability to provide a coherent picture of

hadrons with simple assumptions is of interest as it can serve as a framework to understand all the new data on the $Q^2 < 12 \text{ GeV}^2$ region. Our study aims at exploring the advantages and drawbacks of the different formulations.

The study of form factors making use of relativistic quark models requires a relation between the variables which enter in the representation of the mass operator, \vec{k}_i and spins, and the variables which enter in the vertex and appear in the current. The relation between these two sets of variables depends on the “form of kinematics” being used. The three forms are named as point, instant and front form. They differ from each other in the kinematical subgroup of the Poincaré group. In point form the kinematical subgroup is the full Poincaré group, in instant kinematics it is the group of rotations and translations at a fixed time, while in front form it is the group that leaves invariant the light cone.

Electromagnetic form factors of two-body systems can be defined as certain matrix elements of the electromagnetic current. In point and instant forms, the charge form factor of $S = 0$ mesons can be defined as follows,

$$F_C(Q^2) = \langle 0, \vec{Q}/2 | I^0(0) | 0, -\vec{Q}/2 \rangle_c \quad (1)$$

where I^0 is the time component of the current and \vec{Q} has been taken to be parallel to the z -axis.

In front form, in the $Q^+ = 0$ frame, the charge form factor can be extracted from the “plus” component of the current, $I^+ = n \cdot I$, with $n = \{-1, 0, 0, 1\}$:

$$F_C(Q^2) = \langle 0 | I^+(0) | 0 \rangle, \quad (2)$$

in this case the momentum transfer is taken to be transverse to the z -direction [7].

For $S = 1$ mesons, such as the ρ , we adopt the definition of Ref. [8]. For point and instant forms, we have:

$$\begin{aligned} G_C(Q^2) &= \frac{1}{3} \left[\langle 0, \frac{\vec{Q}}{2} | I^0(0) | -\frac{\vec{Q}}{2}, 0 \rangle_c + 2 \langle 1, \frac{\vec{Q}}{2} | I^0(0) | -\frac{\vec{Q}}{2}, 1 \rangle_c \right], \\ G_M(Q^2) &= \sqrt{\frac{2}{\eta}} \langle 1, \frac{\vec{Q}}{2} | I_+(0) | -\frac{\vec{Q}}{2}, 0 \rangle_c, \\ G_D(Q^2) &= \frac{1}{2\eta} \left[\langle 0, \frac{\vec{Q}}{2} | I^0(0) | -\frac{\vec{Q}}{2}, 0 \rangle_c - \langle 1, \frac{\vec{Q}}{2} | I^0(0) | -\frac{\vec{Q}}{2}, 1 \rangle_c \right], \end{aligned} \quad (3)$$

while for front form,

$$\begin{aligned}
G_C(Q^2) &= F_{0d} + \frac{1}{6}F_{2d} - \frac{2}{3}\eta \left\{ F_{0d} + F_{2d} + \frac{5}{2}F_{1d} \right\}, \\
G_M(Q^2) &= 2F_{0d} + F_{2d} + F_{1d}(1 - \eta), \\
G_D(Q^2) &= \frac{1}{\eta} \left\{ F_{2d} + \eta \left(\frac{1}{2}F_{2d} - F_{0d} - F_{1d} \right) \right\}.
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
F_{0d}(Q^2) &= \frac{1}{2(1 + \eta)} \{ \langle 1 | I^+(0) | 1 \rangle + \langle 0 | I^+(0) | 0 \rangle \}, \\
F_{1d}(Q^2) &= \frac{-\sqrt{2}}{\sqrt{\eta}(1 + \eta)} \langle 1 | I^+(0) | 0 \rangle, \\
F_{2d}(Q^2) &= \frac{-1}{(1 + \eta)} \langle 1 | I^+(0) | -1 \rangle.
\end{aligned} \tag{5}$$

The kinematical variable η is defined as $\eta = \frac{1}{4}(v_f - v_a)^2 = Q^2/4M^2$, where M is the meson mass. In a previous work [5], the momentum appears scaled as $p = \frac{M}{2m_q} \frac{Q}{2}$, which means, $\eta = \frac{Q^2}{16m_q^2}$, where m_q is the mass of quark.

With the definitions in Eqs. (3), (4) and (5), the charge and magnetic and quadrupole moments of spin 1 mesons are defined as,

$$eG_C(0) = e, \quad eG_M(0) = 2M\mu, \quad eG_Q(0) = M^2\mathcal{D}, \tag{6}$$

where e is the electron charge and M is the meson mass.

Meson states are represented by eigenfunctions of the mass operator, which are functions of internal momenta, \vec{k}_i , and spin variables. We use a simple spectral representation of the mass operator, considering only the π and the ρ . The meson wave functions are constructed in the naive quark model [9],

$$\begin{aligned}
\psi^\pi(\vec{q}) &= \xi_c \varphi_0(\vec{q}) \phi_S \chi_A, \\
\psi^\rho(\vec{q}) &= \xi_c \varphi_0(\vec{q}) \phi_A \chi_S,
\end{aligned} \tag{7}$$

where ξ_c is the fully symmetric color wave function. The flavor wave functions $\phi_{S,A}$ have the forms:

$$\begin{aligned}
\phi_{S,A}^+ &= \frac{1}{\sqrt{2}}(u\bar{d} \pm \bar{d}u), \\
\phi_{S,A}^0 &= \frac{1}{2}[(d\bar{d} - u\bar{u}) \pm (\bar{d}d - \bar{u}u)], \\
\phi_{S,A}^- &= -\frac{1}{\sqrt{2}}(d\bar{u} \pm \bar{u}d).
\end{aligned} \tag{8}$$

The spin wave functions, χ , are the usual:

$$\chi_S^1 = \uparrow\uparrow, \quad \chi_{S,A}^0 = \frac{1}{\sqrt{2}}(\uparrow\downarrow \pm \downarrow\uparrow), \quad \chi_S^{-1} = \downarrow\downarrow. \quad (9)$$

The effect of the Lorentz transformation on the spin variables for canonical spins is accounted by a Wigner rotation of the form: $D_{\lambda_i, \sigma_i}^{1/2}(R_W[B(v_K), k_i])$ with $R_W[B(v_K), k_i] := B^{-1}(p_i)B(v_K)B(k_i)$, where $B(v)$ are rotationless Lorentz transformations, and v_K is the boost velocity.

For the spatial part of the wave function, we adopt both Gaussian and rational forms:

$$\varphi_0^G(\vec{q}) = \frac{1}{(b\sqrt{\pi})^{3/2}} e^{-\vec{q}^2/2b^2}, \quad \varphi_0^R(\vec{q}) = \mathcal{N}(1 + \vec{q}^2/2b^2)^{-a}, \quad (10)$$

where $\vec{q} = \frac{1}{\sqrt{2}}(\vec{k}_2 - \vec{k}_1)$ and \mathcal{N} is a normalization constant. In the center of mass frame we have $\vec{k}_1 + \vec{k}_2 = 0$ and thus $\vec{k}_2 = \frac{1}{\sqrt{2}}\vec{q} = -\vec{k}_1$. The Jacobians of the transformation between the variables are:

for point form,

$$J(\vec{v}; \vec{p}_2) := \left(\frac{\partial \vec{q}}{\partial \vec{p}_2} \right)_{\vec{v}} = 2\sqrt{2} \frac{\omega_2}{E_2} = 2\sqrt{2} \frac{(E_2 v^0 - p_{2z} v_z)}{E_2}. \quad (11)$$

for front form

$$J(\mathbf{P}; \mathbf{p}_2) := \left(\frac{\partial \vec{q}}{\partial (\xi_2, \mathbf{k}_{2\perp})} \right)_{\mathbf{P}} = 2\sqrt{2} \frac{\partial k_z}{\partial \xi} = 2\sqrt{2} \frac{M_0}{4\xi(1-\xi)}, \quad (12)$$

with

$$k_{zi} = \frac{1}{2} \left(\xi_i M_0 - \frac{m_q^2 + k_{i\perp}^2}{\xi_i M_0} \right) = M_0(\xi - 1/2),$$

$$M_0^2 = \sum_i \frac{m_q^2 + k_{i\perp}^2}{\xi_i} = \frac{m_q^2 + k_{\perp}^2}{\xi(1-\xi)}. \quad (13)$$

and for instant form,

$$J(\vec{P}, \vec{p}_2) = 2\sqrt{2} \frac{\omega_2}{E_2} \left\{ 1 - \frac{E_2 v_z}{M_0} \left(\frac{p_{1z}}{E_1} - \frac{p_{2z}}{E_2} \right) \right\}, \quad (14)$$

where

$$P_x = P_y = 0, \quad M_0^2 = (\sum_i E_i)^2 - |\vec{P}|^2, \quad \vec{v} := \frac{\vec{P}}{M_0}. \quad (15)$$

For each form of kinematics the dynamics generates the current density operator from a kinematic current. For point form we have,

$$\langle \vec{v}_f, \vec{v}_2' | I^\mu(0) | \vec{v}_2, \vec{v}_a \rangle = \delta^{(3)}(v_2' - v_2) (\frac{1}{6} + \frac{1}{2}\tau_3^{(1)}) \bar{u}(\vec{v}_1') \gamma^{(1)\mu} u(\vec{v}_1), \quad (16)$$

for front form,

$$\begin{aligned} & \langle P^+, P_{\perp f}, \mathbf{p}_2' | I^+(x^-, x_\perp) | \mathbf{p}_2, P_{\perp a}, P^+ \rangle \\ &= \delta^{(3)}(p_2' - p_2) (\frac{1}{6} + \frac{1}{2}\tau_3^{(1)}) \bar{u}(\mathbf{p}_1') \gamma^{(1)+} u(\mathbf{p}_1) e^{i(P_{\perp f} - P_{\perp a}) \cdot x_\perp}, \end{aligned} \quad (17)$$

and for instant form,

$$\begin{aligned} & \langle \frac{1}{2}\vec{Q}, \vec{p}_2' | I^\mu(\vec{x}) | \vec{p}_2, -\frac{1}{2}\vec{Q} \rangle \\ &= \delta^{(3)}(p_2' - p_2) (\frac{1}{6} + \frac{1}{2}\tau_3^{(1)}) \bar{u}(\vec{p}_1') \gamma^{(1)\mu} u(\vec{p}_1) e^{i(\vec{Q} \cdot \vec{x})}. \end{aligned} \quad (18)$$

With the formulas given above, we can calculate the form factors of the π and the ρ . The procedure used to fix the meson states is the following. We fix a , b and m_q (or just b and m_q for the gaussian case) so that they are both in the range of other similar calculations and that the π form factor and charge radius are fairly reproduced. The use of two different wave functions allows us to estimate the theoretical uncertainty derived from the wave function used.

The first relevant issue we notice is that it is not possible to find a set of parameters with any of the wave functions in point form so that the Q^2 behavior of the form factor is reproduced. This was one of the points raised in Ref. [4]. For instant and front forms it is possible to find such a set of parameters using both types of wave functions. The sets of parameters are given in Table 1.

In Fig. 1 the π form factor obtained with the parameters of Table 1 is presented. The bands depicted in the figures are constructed using the results obtained with the gaussian and rational wave functions, one gives the band minimum while the other provides the maximum. In this way the band gives an estimate of the theoretical uncertainty due to the specific choice of wave function. The chosen parameter sets permit a good reproduction of the Q^2 behavior of the data in instant and front forms. The charge radii calculated in front and instant forms are quite close to the experimental data, small discrepancies with the data could be attributed to our simple model were known

	b [MeV]	m_q [MeV]	a	$\sqrt{\langle r_\pi^2 \rangle}$ [fm]
Gaussian				
Instant form	370	140	—	0.600
Point form	3000	380	—	3.018
Front form	450	250	—	0.665
Rational				
Instant form	700	150	5	0.619
Point form	3000	300	1	2.545
Front form	600	250	3	0.659

Table 1

Parameters and charge radius of the π in instant, point and front form both for the rational and gaussian spatial wave functions. The experimental value for the charge radius is $\sqrt{\langle r_\pi^2 \rangle} = 0.663 \pm 0.006$ fm [10].

effects arising from vector meson contributions to the charge radius of the pion are not accounted for [11]. The result obtained with point form is completely off and cannot be brought into agreement by changing the parameters of the model wave functions.

The high Q^2 behavior of the form factor is qualitatively similar in instant and front form although the instant form result falls slower. In both cases the falloff of the form factor at large Q^2 is faster than the QCD predictions of Refs. [12,13], $Q^2 F(Q^2) \propto \text{const}$ or $(1/\log Q^2)$. In fact the obtained behavior is closer to $Q^2 F(Q^2) \propto 1/Q^2$. This faster falloff, of almost one power of Q^2 , seems to be a general trend in most quantum mechanical calculations where the coupling of the photon to the standard quark current is considered [14]. Improvements, e.g. considering two-body currents or different quark-photon couplings, are beyond the scope of this letter.

We have shown that the π form factor can be reasonably understood with instant and front form of kinematics by finding the appropriate mass operator, which in our case corresponded to finding the parameters of Table 1. Now we consider the case of the ρ meson. Due to the fact that the parameters for point form could not be fixed from the pion charge form factor, we chose them as similar to those of Ref. [5] (let us note that for the ρ the prescription used in their paper is irrelevant due to the fact that $\rho \approx 2m_q$). In Table 2 the values for the magnetic and quadrupole moments defined in Eq. (6) are presented. The results for the ρ magnetic moment in all cases are smaller than $2 e/2M_\rho$, and also smaller than other theoretical estimates. The quadrupole moments obtained, which ranges between $[0.2 - 0.5] e/M_\rho^2$, are consistent with Refs. [17,18].

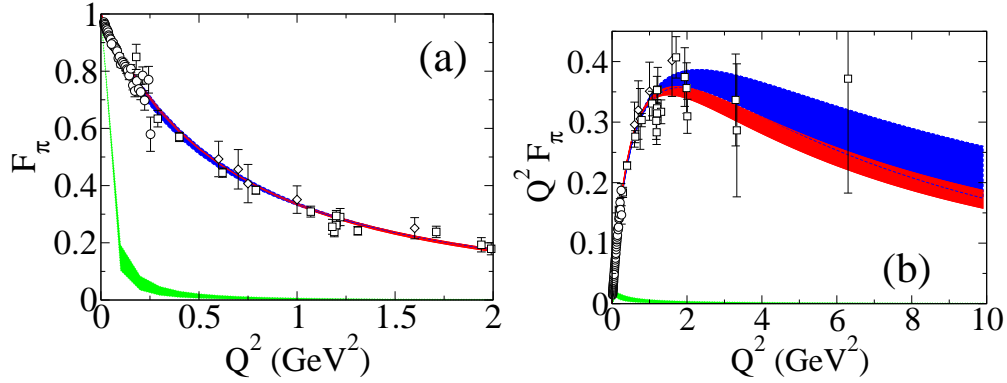


Fig. 1. (a) π charge form factor as function of $Q^2(\text{GeV}^2)$. The band is obtained as explained in the text. Red, green and blue stand for instant, point and front form. (b) Same as (a) but multiplied by Q^2 . The experimental data are from Refs. [10,15,16]

	$\mu [e/2M_\rho]$	$\mathcal{D} [e/M_\rho^2]$
Instant form	1.5	[0.36–0.29]
Point form	0.9	[0.38–0.50]
Front form	1.5	[0.2–0.33]
Choi <i>et al.</i> [17]	1.9	0.43
Jaus [18]	1.83	0.33
Cardarelli <i>et al.</i> [3]	2.23	0.61

Table 2

Magnetic and quadrupole moments of the ρ for instant, point, and front form. The range correspond to using gaussian or rational wave functions.

In Figs. 2, 3 and 4 we present the Q^2 dependence of the electromagnetic form factors defined in Eqs. (3) and (4).

In Fig 2 the charge form factor is depicted. Unlike in the case of the π it can be seen that in this case the three forms provide a coherent picture in the low Q^2 region. However, in the high- Q^2 region the situation is different. Essentially, point and instant form predict a behavior close to the one observed in the pion charge form factor, while front form falls faster and eventually crosses zero at $Q^2 \approx 4.5 \text{ GeV}^2$. This feature of the charge form factor becoming negative in front form calculations is also present in the electric form factor of the proton, see Ref. [7], and in other front-form calculations of the ρ charge form factors [2,3]. The Q^2 dependence of the form factors at high Q^2 is mostly independent of the wave function used as can be easily seen by the thinness of the bands.

The failure of point form to reproduce the π form factor is therefore most likely

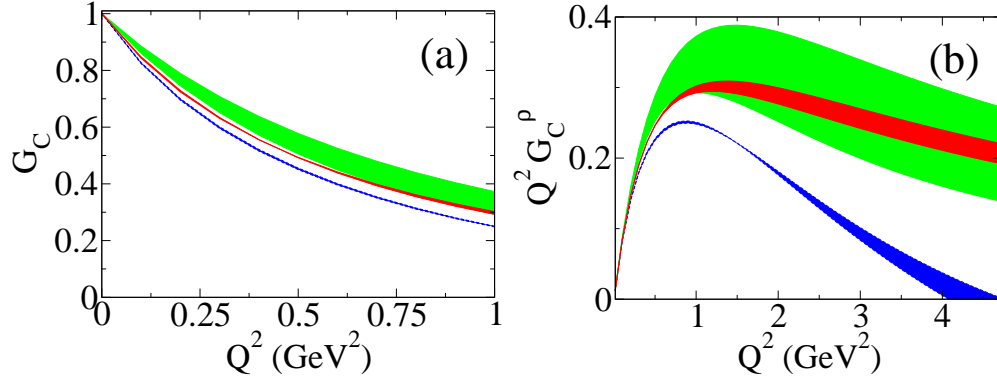


Fig. 2. Charge form factor of the ρ as function of $Q^2(\text{GeV}^2)$. Same description as Fig. 1.

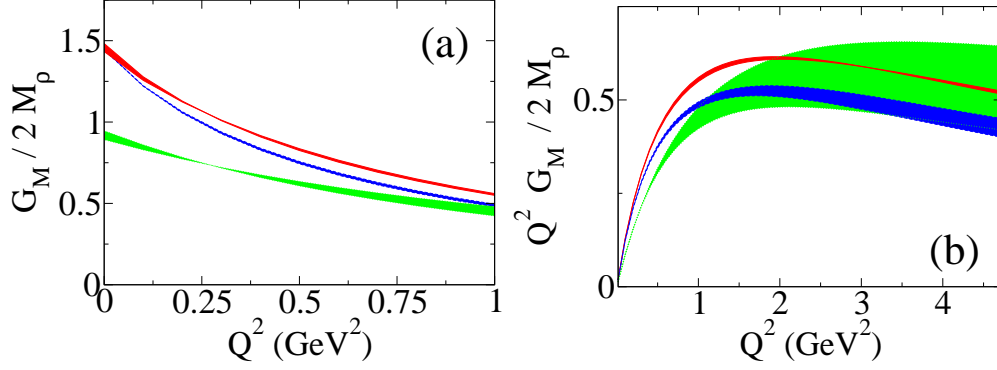


Fig. 3. Magnetic form factor of the ρ over $2M_\rho$ as function of $Q^2(\text{GeV}^2)$. Same description as Fig. 1.

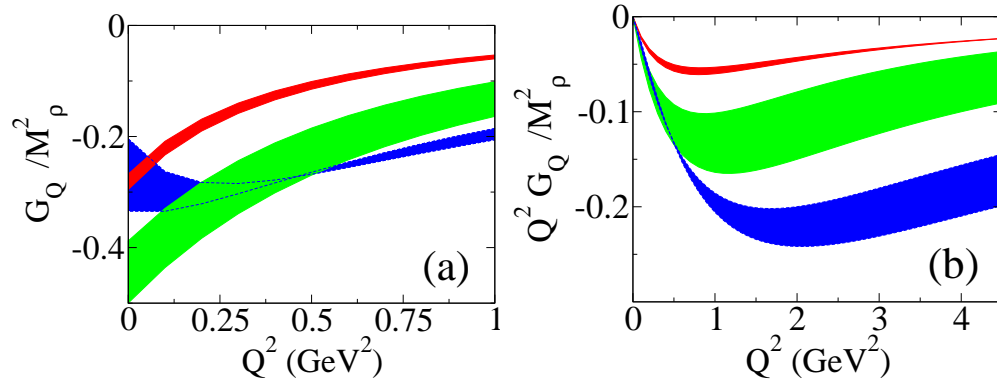


Fig. 4. Quadrupolar form factor of the ρ over M_ρ^2 as function of $Q^2(\text{GeV}^2)$. Same description as Fig. 1.

due to the small mass of the π , as explained in Ref. [19]. On the other hand our study shows that for higher mass mesons, such as the ρ , it is possible to find

an appropriate mass operator such that point form gives qualitatively similar results to front or instant form, leaving the case of the π as a pathological one.

The magnetic form factor is shown in Fig. 3. In this case instant and front form predict a similar magnetic moment, which is given in Table 2, while point form predicts a magnetic moment which is 30% lower. The high Q^2 behavior is similar for the three forms and is compatible with $\propto 1/Q^2$.

Similar situation, but in this case with point form providing a larger value, appears in the quadrupole form factor, which is given in Fig. 4. Instant and front forms give similar quadrupole magnetic moments although with a very different prediction for the Q^2 dependence of the form factor.

We have studied the electromagnetic form factors of the π and the ρ making use of the three different forms of relativistic kinematics. Front and instant forms provide a correct picture of the π electromagnetic form factor, giving both the correct charge radius and Q^2 dependence. The high- Q^2 dependence predicted for the charge form factor of the pion is faster than the one predicted from QCD calculations. Both front and instant forms give similar results for the electromagnetic form factors of the ρ . Point form does not allow a description of the π electric form factor, most likely due to its small mass, and, although qualitatively similar, gives different quantitative values for the ρ electromagnetic form factors. Our calculated values for the ρ dipole and quadrupole moment are around 20% smaller than the ones available in the literature.

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